

Vectorial Lie superalgebras and orthosymplectic decomposition

Marcus K. McLaurin

PhD Student

Industrial & Computational Mathematics

3 October 2026

Advisors:

Asamoah Nkwanta and

Dwight Anderson Williams II



Motivation

Let $\mathfrak{t}$ be a finite-dimensional Lie superalgebra.

Motivation

Let $\mathfrak{t}$ be a finite-dimensional Lie superalgebra.

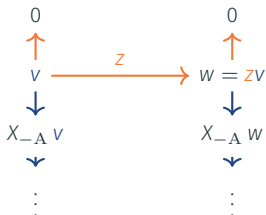
For $\phi: \mathfrak{osp}(1|2) \rightarrow \mathfrak{t}$ there is an associated reduction superalgebra $Z = Z(\phi)$.

Motivation

Let $\mathfrak{t}$ be a finite-dimensional Lie superalgebra.

For $\phi: \mathfrak{osp}(1|2) \rightarrow \mathfrak{t}$ there is an associated reduction superalgebra $Z = Z(\phi)$.

Each z in Z sends a singular vector v to a singular vector $w = zv$.



General Application

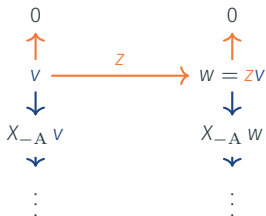
A reduction superalgebra Z aids in decomposing $\mathfrak{t}$ -modules as $\mathfrak{osp}(1|2)$ -modules, providing explicit bases in the process.

Motivation

Let $\mathfrak{t}$ be a finite-dimensional Lie superalgebra.

For $\phi: \mathfrak{osp}(1|2) \rightarrow \mathfrak{t}$ there is an associated reduction superalgebra $Z = Z(\phi)$.

Each z in Z sends a singular vector v to a singular vector $w = zv$.



General Application

A reduction superalgebra Z aids in decomposing $\mathfrak{t}$ -modules as $\mathfrak{osp}(1|2)$ -modules, providing explicit bases in the process.

Research Question

Consider an embedding ϕ of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ in the vectorial Lie superalgebra $\mathfrak{vect}(0|2)$. How can we present the genetics of a reduction superalgebra $Z(\phi)$?

The category of super vector spaces

We work in $\text{sVec}\mathbb{C}$. Our objects are $\mathbb{Z}/2\mathbb{Z}$ -graded: $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

The category of super vector spaces

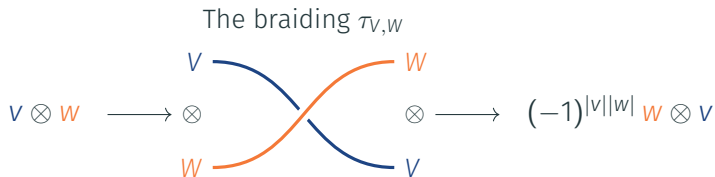
We work in $\text{sVec}\mathbb{C}$. Our objects are $\mathbb{Z}/2\mathbb{Z}$ -graded: $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

The braiding $\tau_{V,W}$

$$V \otimes W \longrightarrow \otimes \xrightarrow{\tau_{V,W}} \otimes \longrightarrow (-1)^{|V||W|} W \otimes V$$

The category of super vector spaces

We work in $\text{sVec}\mathbb{C}$. Our objects are $\mathbb{Z}/2\mathbb{Z}$ -graded: $V = V_{\bar{0}} \oplus V_{\bar{1}}$.



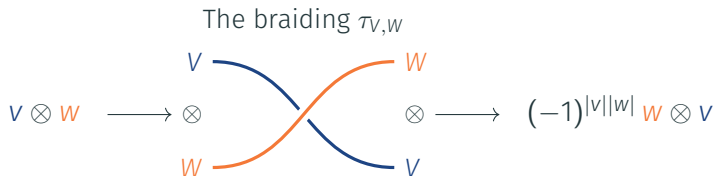
For a morphism $\mu: V \otimes V \rightarrow V$, the braiding determines

(skew)symmetry $\mu = \pm \mu \circ \tau$ and

Jacobi identity $\mu \circ \text{Id} \otimes \mu = \mu \circ \mu \otimes \text{Id} + \mu \circ \text{Id} \otimes \mu \circ \tau \otimes \text{Id}$ on $V^{\otimes 3}$.

The category of super vector spaces

We work in $\text{sVec}\mathbb{C}$. Our objects are $\mathbb{Z}/2\mathbb{Z}$ -graded: $V = V_{\bar{0}} \oplus V_{\bar{1}}$.



For a morphism $\mu: V \otimes V \rightarrow V$, the braiding determines

(skew)symmetry $\mu = \pm \mu \circ \tau$ and

Jacobi identity $\mu \circ \text{Id} \otimes \mu = \mu \circ \mu \otimes \text{Id} + \mu \circ \text{Id} \otimes \mu \circ \tau \otimes \text{Id}$ on $V^{\otimes 3}$.

A super vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ paired with a skewsymmetric μ satisfying the Jacobi identity is a Lie superalgebra.

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$.

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$
- Choice basis for $\mathbf{vect}(0|2)$:
 $\mathcal{B} = \{H_A, H_B, X_{2A}, X_{-2A}; X_A, X_{-A}, X_B, X_{-B}\}$

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$
- Choice basis for $\mathbf{vect}(0|2)$:
 $\mathcal{B} = \{H_A, H_B, X_{2A}, X_{-2A}; X_A, X_{-A}, X_B, X_{-B}\}$

Further:

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$
- Choice basis for $\mathbf{vect}(0|2)$:
 $\mathcal{B} = \{H_A, H_B, X_{2A}, X_{-2A}; X_A, X_{-A}, X_B, X_{-B}\}$

Further:

- Positive and negative subalgebras $\mathfrak{g}_{\pm} = \mathbb{C}X_{\pm A} \oplus \mathbb{C}X_{\pm 2A}$.

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$
- Choice basis for $\mathbf{vect}(0|2)$:
 $\mathcal{B} = \{H_A, H_B, X_{2A}, X_{-2A}; X_A, X_{-A}, X_B, X_{-B}\}$

Further:

- Positive and negative subalgebras $\mathfrak{g}_{\pm} = \mathbb{C}X_{\pm A} \oplus \mathbb{C}X_{\pm 2A}$.
- Cartan subalgebra $\mathfrak{h} = \mathbb{C}H_A$.

The Lie superalgebras we need

The vectorial Lie superalgebras come from supergeometry: The set of derivations on the function algebra $\Lambda(\theta_1, \theta_2)$ is $\mathfrak{t} = \mathbf{vect}(0|2)$. The orthosymplectic Lie superalgebra $\mathfrak{g} = \mathbf{osp}(1|2)$ is a Lie subsuperalgebra of $\mathfrak{t}$.

Important facts:

- Indeterminates: $\theta_1\theta_2 = -\theta_2\theta_1$ (commute in $\mathbf{sVec}\mathbb{C}$)
- Standard basis for $\Lambda(\theta_1, \theta_2)$ is $\{1, \theta_1\theta_2; \theta_1, \theta_2\}$
- Choice basis for $\mathbf{vect}(0|2)$:
 $\mathcal{B} = \{H_A, H_B, X_{2A}, X_{-2A}; X_A, X_{-A}, X_B, X_{-B}\}$

Further:

- Positive and negative subalgebras $\mathfrak{g}_{\pm} = \mathbb{C}X_{\pm A} \oplus \mathbb{C}X_{\pm 2A}$.
- Cartan subalgebra $\mathfrak{h} = \mathbb{C}H_A$.
- Known decomposition $\mathfrak{t} = (\mathbb{C}X_{-B} \oplus \mathbb{C}H_B \oplus \mathbb{C}X_B) \oplus \mathfrak{g}$.

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

Let $U = U(\mathfrak{t})$.

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

Let $U = U(\mathfrak{t})$. The reduction superalgebra $Z = Z(\phi)$ is $\text{Norm}_U(U\mathfrak{g}_+)/U\mathfrak{g}_+$. Write $Z = N/I_+$.

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

Let $U = U(\mathfrak{t})$. The reduction superalgebra $Z = Z(\phi)$ is $\text{Norm}_U(U\mathfrak{g}_+)/U\mathfrak{g}_+$. Write $Z = N/I_+$.

Two issues: not a finitely generated $\mathbb{C}$ -algebra and hard to describe explicit elements of N .

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

Let $U = U(\mathfrak{t})$. The reduction superalgebra $Z = Z(\phi)$ is $\text{Norm}_U(U\mathfrak{g}_+)/U\mathfrak{g}_+$. Write $Z = N/I_+$.

Two issues: not a finitely generated $\mathbb{C}$ -algebra and hard to describe explicit elements of N .

Reminder of motivation: How can we describe the generators and relations of Z ? A nice consequence is that we would know how Z acts on the subspace of singular vectors within certain $\mathfrak{t}$ -representations.

The reduction superalgebra for ϕ

To each Lie superalgebra we have the universal enveloping superalgebra:

$$\{\mathfrak{g} \rightarrow \mathfrak{gl}(V)\} \leftrightarrow \{U(\mathfrak{g}) \rightarrow \text{End}(V)\}.$$

Let $U = U(\mathfrak{t})$. The reduction superalgebra $Z = Z(\phi)$ is $\text{Norm}_U(U\mathfrak{g}_+)/U\mathfrak{g}_+$. Write $Z = N/I_+$.

Two issues: not a finitely generated $\mathbb{C}$ -algebra and hard to describe explicit elements of N .

Reminder of motivation: How can we describe the generators and relations of Z ? A nice consequence is that we would know how Z acts on the subspace of singular vectors within certain $\mathfrak{t}$ -representations.

Resolution: We localize Z to present Z as a finitely generated ring. We utilize the extremal projector

$$P = \sum_{n \geq 0} \frac{(-1)^n}{[n/2]! \prod_{k=1}^{\lceil n/2 \rceil} (H_A + k)} X_{-A}^n X_A^n.$$

The diamond product table

More details on Z and the so-called diamond product are in [Zhe89; KO10; HW22; HW25], my dissertation work (in progress), and forthcoming manuscripts.

The diamond product table

More details on Z and the so-called diamond product are in [Zhe89; KO10; HW22; HW25], my dissertation work (in progress), and forthcoming manuscripts.

Theorem (Genetics of $Z(\phi)$)

Over a localization of $U(\mathfrak{h})$ we have the following product table describing the generators and relations of Z .

The diamond product table

More details on Z and the so-called diamond product are in [Zhe89; KO10; HW22; HW25], my dissertation work (in progress), and forthcoming manuscripts.

Theorem (Genetics of $Z(\phi)$)

Over a localization of $U(\mathfrak{h})$ we have the following product table describing the generators and relations of Z .

$\diamond$	$g(H_A)$	X_{-B}	H_B	X_B
$f(H_A)$	$f(H_A)g(H_A)$	$f(H_A)X_{-B}$	$f(H_A)H_B$	$f(H_A)X_B$
X_{-B}	$g(H_A + 1)X_{-B}$	0	$X_{-B} \diamond H_B$	$X_{-B} \diamond X_B$
H_B	$g(H_A)H_B$	$\frac{H_A}{H_A+1} X_{-B} \diamond H_B$	$H_B \diamond H_B$	$H_B \diamond X_B$
X_B	$g(H_A - 1)X_B$	$-\frac{H_A}{H_A+1} X_{-B} \diamond X_B$ $+ \frac{1}{H_A} H_B \diamond H_B - H_A$	$\frac{H_A-1}{H_A} H_B \diamond X_B$	0

The present work is supported by

- NSF Award #2503008
 - Title: Excellence in Research: Super Representation Theory, Quantum Spaces, and Algebra Maps
 - PI: Dwight Anderson Williams II

The present work is supported by

- NSF Award #2503008
 - Title: Excellence in Research: Super Representation Theory, Quantum Spaces, and Algebra Maps
 - PI: Dwight Anderson Williams II
- Wonderful audience
 - Thank you! Questions?

References

- [HW22] J. T. Hartwig and D. A. Williams II. “**Diagonal reduction algebra for $\mathfrak{osp}(1|2)$** ”. *Theoretical and Mathematical Physics* 210.2 (Feb. 2022), pp. 155–171. ISSN: 0040-5779, 1573-9333. DOI: [10.1134/S0040577922020015](https://doi.org/10.1134/S0040577922020015). URL: <https://link.springer.com/10.1134/S0040577922020015>.
- [HW25] J. T. Hartwig and D. A. Williams II. “**Symplectic Differential Reduction Algebras and Generalized Weyl Algebras**”. *Symmetry, Integrability and Geometry: Methods and Applications* 21.001 (Jan. 2025). ISSN: 18150659. DOI: [10.3842/SIGMA.2025.001](https://doi.org/10.3842/SIGMA.2025.001). URL: <https://www.emis.de/journals/SIGMA/2025/001/>.

- [KO10] S. M. Khoroshkin and O. V. Ogievetsky. **“Diagonal reduction algebras of gl type”**. *Functional Analysis and Its Applications* 44.3 (Sept. 2010), pp. 182–198. ISSN: 0016-2663, 1573-8485. DOI: 10.1007/s10688-010-0023-0. URL: <http://link.springer.com/10.1007/s10688-010-0023-0>.
- [Zhe89] D. P. Zhelobenko. **“Extremal Projectors and Generalized Mickelsson Algebras over Reductive Lie Algebras”**. *Mathematics of the USSR-Izvestiya* 33.1 (Feb. 1989), pp. 85–100. ISSN: 0025-5726. DOI: 10.1070/IM1989v033n01ABEH000815. URL: <http://stacks.iop.org/0025-5726/33/i=1/a=A04?key=crossref.e128096c29ea8e351ed1ccd3a90c9474>.

The bracket table of $\mathfrak{vect}(0|2)$

$[\cdot, \cdot]$	H_A	H_B	X_{2A}	X_{-2A}	X_A	X_{-A}	X_B	X_{-B}
H_A	0	0	$2X_{2A}$	$-2X_{-2A}$	X_A	$-X_{-A}$	X_B	$-X_{-B}$
H_B	0	0	0	0	X_B	X_{-B}	X_A	X_{-A}
X_{2A}	$-2X_{2A}$	0	0	H_A	0	$-X_A$	0	$-X_B$
X_{-2A}	$2X_{-2A}$	0	$-H_A$	0	$-X_{-A}$	0	$-X_{-B}$	0
X_A	$-X_A$	$-X_B$	0	X_{-A}	$2X_{2A}$	H_A	0	H_B
X_{-A}	X_{-A}	$-X_{-B}$	X_A	0	H_A	$-2X_{-2A}$	$-H_B$	0
X_B	$-X_B$	$-X_A$	0	X_{-B}	0	$-H_B$	$-2X_{2A}$	$-H_A$
X_{-B}	X_{-B}	$-X_{-A}$	X_B	0	H_B	0	$-H_A$	$2X_{-2A}$

The adjoint module of $\mathfrak{osp}(1|2)$

$[\cdot, \cdot]$	H_A	X_{2A}	X_{-2A}	X_A	X_{-A}
H_A	0	$2X_{2A}$	$-2X_{-2A}$	X_A	$-X_{-A}$
X_{2A}	$-2X_{2A}$	0	H_A	0	$-X_A$
X_{-2A}	$2X_{-2A}$	$-H_A$	0	$-X_{-A}$	0
X_A	$-X_A$	0	X_{-A}	$2X_{2A}$	H_A
X_{-A}	X_{-A}	X_A	0	H_A	$-2X_{-2A}$

The standard module of $\mathfrak{osp}(1|2)$

$[\cdot, \cdot]$	H_B	X_B	X_{-B}
H_A	0	X_B	$-X_{-B}$
X_{2A}	0	0	$-X_B$
X_{-2A}	0	$-X_{-B}$	0
X_A	$-X_B$	0	H_B
X_{-A}	$-X_{-B}$	$-H_B$	0

Brackets of the original even elements

$[\cdot, \cdot]$	$\theta_1 \partial_1$	$\theta_1 \partial_2$	$\theta_2 \partial_1$	$\theta_2 \partial_2$
$\theta_1 \partial_1$	0	$\theta_1 \partial_2$	$-\theta_2 \partial_1$	0
$\theta_1 \partial_2$	$-\theta_1 \partial_2$	0	$\theta_1 \partial_1 - \theta_2 \partial_2$	$\theta_1 \partial_2$
$\theta_2 \partial_1$	$\theta_2 \partial_1$	$\theta_2 \partial_2 - \theta_1 \partial_1$	0	$-\theta_2 \partial_1$
$\theta_2 \partial_2$	0	$-\theta_1 \partial_2$	$\theta_2 \partial_1$	0

Brackets of the original odd elements

$[\cdot, \cdot]$	∂_1	∂_2	$\theta_1\theta_2\partial_1$	$\theta_1\theta_2\partial_2$
∂_1	0	0	$\theta_2\partial_1$	$\theta_2\partial_2$
∂_2	0	0	$-\theta_1\partial_1$	$-\theta_1\partial_2$
$\theta_1\theta_2\partial_1$	$\theta_2\partial_1$	$-\theta_1\partial_1$	0	0
$\theta_1\theta_2\partial_2$	$\theta_2\partial_2$	$-\theta_1\partial_2$	0	0

The alternative basis as derivations

$$H_A = -\theta_1\partial_1 + \theta_2\partial_2, \quad H_B = \theta_1\partial_1 + \theta_2\partial_2,$$

$$X_{2A} = \theta_2\partial_1, \quad X_{-2A} = \theta_1\partial_2,$$

$$X_A = \theta_1\theta_2\partial_1 + \partial_1, \quad X_B = \theta_1\theta_2\partial_1 - \partial_1,$$

$$X_{-A} = \theta_1\theta_2\partial_2 + \partial_2, \quad X_{-B} = \theta_1\theta_2\partial_2 - \partial_2.$$