

## Motivation

In representation theory, we consider the reduction problem: How do we decompose a simple representation of a Lie superalgebra as a representation of a Lie subsuperalgebra? The present work describes the technique of using reduction algebras to address the reduction problem. The results provide traction in the classification of simple representations and aids in computational use of representation theory within mathematical physics. Additionally, the determination of a reduction algebra in terms of generators and relations leads to noncommutative ring theory and examples of quantum-type or dynamical algebras. We are most interested in cases of reducing representations of vectorial Lie superalgebras (particularly Lie superalgebras of superderivations on exterior algebras), especially since complete presentations of reduction superalgebras are rarely provided in the literature [HW22].

The route of our work begins with a recalling of definitions and examples of Lie superalgebras, the definition of a reduction superalgebra associated to an associative superalgebra homomorphism, and a discussion on extremal projectors. We end with future steps towards defining a co-rank one orthosymplectic reduction superalgebra, which is associated to an embedding of the orthosymplectic Lie superalgebra  $\mathfrak{osp}(1|2)$  in the vectorial Lie superalgebra  $\mathfrak{vect}(0|2) \cong \mathfrak{osp}(2|2)$ .

## The category $\mathbf{sVec}$ of super vector spaces

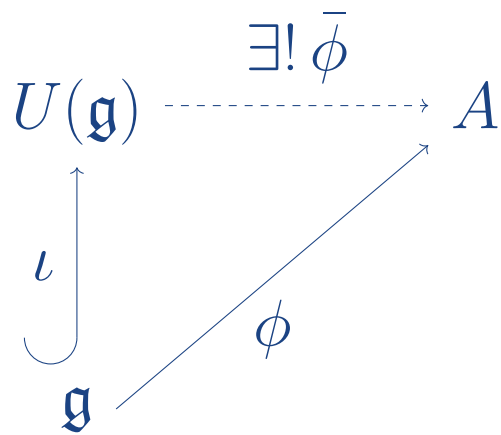
We reference [Rie16; Wal24] for categorical notions. Let  $\mathbb{Z}/2\mathbb{Z}$  be the field  $\{\bar{0}, \bar{1}\}$  of two elements. The objects of  $\mathbf{sVec}$  are  $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  called super vector spaces with morphisms as parity-preserving linear maps. The Lie algebra objects are pairs formed by an object  $\mathfrak{g}$  and a skew-symmetric morphism  $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$  satisfying the Jacobi identity. All notions of symmetry are governed by the super braiding  $\tau: v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$ . For example, the commutator for algebra objects with a multiplication  $\mu$  appears as  $\mu - \mu \circ \tau$ .

## Linear Lie superalgebras

Let  $V$  be a super vector space with  $\dim V_{\bar{0}} = m$  and  $\dim V_{\bar{1}} = n$ . The space  $\text{End}(V)$  of all liner maps is a unital associative superalgebra, and the pair  $(\text{End}(V), [\cdot, \cdot])$  forms the general linear Lie superalgebra  $\mathfrak{gl}(V)$  of  $V$ , which is isomorphic to  $(\mathfrak{gl}(m|n), [\cdot, \cdot])$ , the general linear Lie superalgebra of degree “ $m, n$ ”. Lie subalgebras of general linear Lie superalgebras are called linear Lie superalgebras. The orhosymplectic Lie superlagbera  $\mathfrak{osp}(m|n)$  is a linear Lie superalgebra.

The bracket-preserving linear maps  $\phi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  are called representations of  $\mathfrak{g}$ .

## Universal property



The question: Is there a most general associative superalgebra in which  $\mathfrak{g}$  embeds? The answer: The pair of the universal enveloping Lie superalgebra  $U(\mathfrak{g})$  and the embedding  $\iota$  from the tensor algebra construction. Note:  $U(\mathfrak{g}) = T(\mathfrak{g}) / \langle xy - (-1)^{|x||y|} yx - [x, y] \rangle$ .

## Reduction algebra techniques using the simple Lie superalgebra $\mathfrak{osp}(1|2)$

The  $(3|2)$ -dimensional Lie superalgebra  $\mathfrak{g} = \mathfrak{osp}(1|2)$  has a matrix representation

$$\mathfrak{osp}(1|2) = \mathfrak{osp}(1|2)_{\bar{0}} \oplus \mathfrak{osp}(1|2)_{\bar{1}} = \left( \mathbb{C}x_{-2\alpha} \oplus \mathbb{C}h \oplus \mathbb{C}x_{2\alpha} \right) \oplus \left( \mathbb{C}x_{-\alpha} \oplus \mathbb{C}x_{\alpha} \right),$$

where  $x_{-2\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $x_{-\alpha} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ,  $h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ,  $x_{\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ ,  $x_{2\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ .

Let  $\mathfrak{g}_{\pm} = \mathbb{C}x_{\pm 2\alpha} \oplus \mathbb{C}x_{\pm \alpha}$  and  $\mathfrak{h} = \mathbb{C}h$ . A triangular decomposition of  $\mathfrak{g}$  is given by  $\mathfrak{g} = \mathfrak{g}_{-} \oplus \mathfrak{h} \oplus \mathfrak{g}_{+}$ .

### Associative superalgebra homomorphism

There exists an embedding  $\mathfrak{g} \hookrightarrow A = U(\mathfrak{osp}(2|2))$ . Therefore we have a map  $\varphi: U(\mathfrak{g}) \rightarrow A$  of unital associative superalgebras. Thus  $A$  is a  $U(\mathfrak{g})$ -bimodule via left and right multiplication.

$$\mathfrak{g}_{+} \subset \mathfrak{g} \xrightarrow{\iota} U(\mathfrak{g}) \xrightarrow{\varphi} A \rightarrow \text{End}(A)$$

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$$\mathfrak{g}_{-} \subset \mathfrak{g} \xrightarrow{\iota} U(\mathfrak{g}) \xrightarrow{\varphi} A \rightarrow \text{End}(A)$$

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### Universal Verma module (with respect to $\mathfrak{g}_{+}$ )

Let  $I_{+} = A\mathfrak{g}_{+}$  be the left ideal of  $A$  generated by the action of  $\mathfrak{g}_{+}$  acting on  $A$  from the right. The left  $A$ -module  $A/I_{+}$  is the universal Verma module.

### Nonlocalized reduction algebra [Zhe06]

Let  $N = N_A(I_{+})$  be the normalizer of  $I_{+}$ , that is, the largest subalgebra of  $A$  in which  $I_{+}$  is a two-sided ideal. Then the subspace  $(A/I_{+})^{+}$  of  $\mathfrak{g}_{+}$ -invariants within  $A/I_{+}$  is the quotient algebra  $N/I_{+}$ , which we call the nonlocalized reduction algebra associated to  $\varphi$ .

Sketch of proof:  $x(a + I_{+}) = xa + I_{+} = 0 + I_{+}$  for all  $x \in \mathfrak{g}_{+}$  if and only if  $xa \in I_{+}$  for all  $x \in \mathfrak{g}_{+}$  if and only if  $a \in N$ .

### Importance of reduction algebra

Let  $V^{+} \subset V$  be the  $\mathfrak{g}_{+}$  invariants. Since  $I_{+}V^{+} \subset \{0\}$  and  $NV^{+} \subset V^{+}$ , we have the action  $(n + I)v = nv$ . So  $V^{+}$  is a left module over the reduction algebra and we can generate  $\mathfrak{g}$ -submodules of  $V$ :

$$U(\mathfrak{g})(n + I_{+})v \subset V \text{ for } n \in N.$$

## Normalizers may be hard to compute systematically

Assume our spaces are localized appropriately. Let  $I_{-} := \mathfrak{g}_{-}A$ . We consider the double coset space  $A/\mathbb{I} := I_{-} \backslash A/I_{+}$ . There exists a unique element  $P$  of a certain completion of  $U(\mathfrak{g})$  that induces a projection  $P_V$  on each  $V$ ; moreover, we have

$$P_{A/I}: A/I \rightarrow A/\mathbb{I} \rightarrow N/I \rightarrow A/I$$

- $\mathfrak{g}_{+}P = 0$ ;
- $P\mathfrak{g}_{-} = 0$ ;
- $[h, P] = 0$ ;
- $P \equiv 1 \pmod{\mathbb{I}}$ ;
- $A/\mathbb{I}$  is a superalgebra under the diamond product [Zhe89].

## Future study

### Applications in noncommutative ring theory and in representation theory

- “Ghost center and representations of the diagonal reduction algebra of  $\mathfrak{osp}(1|2)$ ” [HW23]

### Extremal projectors

- “Fortieth Anniversary of Extremal Projector Method” [Tol05]

### Solving equations in mathematical physics

- Dirac Reduction Algebra* [DHW25]
- Exact Solutions to the Klein–Gordon Equation via Reduction Algebras* [HUW25]

### Understanding of quantum algebras and dynamical quantum groups

- “Mickelsson Algebras and Zhelobenko Operators” [KO08]
- “R-Matrix and Mickelsson Algebras for Orthosymplectic Quantum Groups” [AM15]

## Acknowledgments

This material is based upon work supported by the NSF under Award No. 2503008.

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