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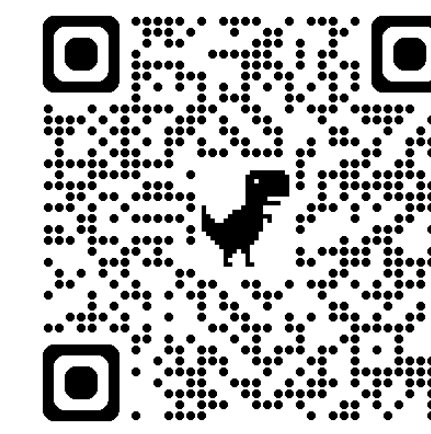


Corank-One Orthosymplectic Reduction Superalgebras

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Motivation

In representation theory, we consider the reduction problem: How do we decompose a simple representation of a Lie superalgebra as a representation of a Lie subsuperalgebra? The present work describes the technique of using reduction algebras to address the reduction problem. The results provide traction in the classification of simple representations and aids in computational use of representation theory within mathematical physics. Additionally, the determination of a reduction algebra in terms of generators and relations furthers noncommutative ring theory and provides examples of quantum-type algebras [Kho04]. We are most interested in cases of reducing representations of vectorial Lie superalgebras, particularly, Lie superalgebras of superderivations on exterior algebras—we specialize to the corank-one case here. We note that [HW22] gives the genetics of a diagonal-type reduction superalgebra, and the authors claim that such complete presentations of reduction superalgebras are not commonly found in the literature.

Progress

Our route includes a recalling of definitions and examples of Lie superalgebra, associating a reduction superalgebras to an embedding of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ in the vectorial Lie superalgebra $\mathfrak{vect}(0|2) \cong \mathfrak{osp}(2|2)$, discussing extremal projectors, and providing a table of diamond products. Future steps generalize corank-one orthosymplectic reduction superalgebras to higher dimensions.

The category $\mathbf{sVec}$ of super vector spaces

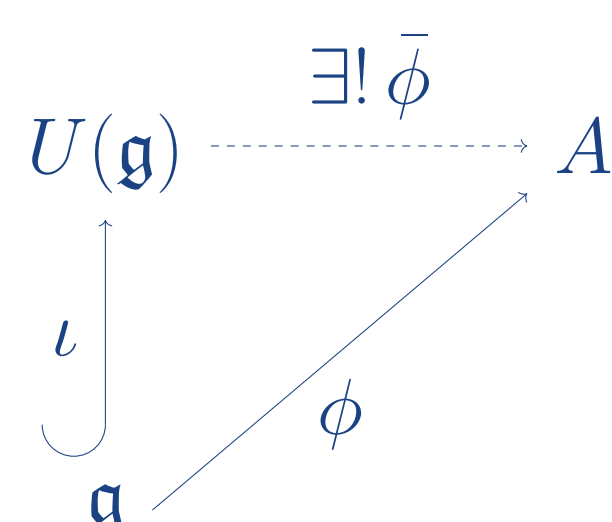
We reference [Rie16; Wal24] for categorical notions. Let $\mathbb{Z}/2\mathbb{Z}$ be the field $\{\bar{0}, \bar{1}\}$ of two elements. The objects of $\mathbf{sVec}$ are $\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$ called super vector spaces with morphisms as parity-preserving linear maps. The Lie algebra objects are pairs formed by an object $\mathfrak{g}$ and a skew-symmetric morphism $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying the Jacobi identity. All notions of symmetry are governed by the super braiding $\tau: v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$. For example, the commutator for algebra objects with a multiplication μ appears as $\mu - \mu \circ \tau$.

Linear Lie superalgebras

Let V be a super vector space with $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. The space $\text{End}(V)$ of all linear maps is a unital associative superalgebra, and the pair $(\text{End}(V), [\cdot, \cdot])$ forms the general linear Lie superalgebra $\mathfrak{gl}(V)$ of V , which is isomorphic to $(\mathfrak{gl}(m|n), [\cdot, \cdot])$, the general linear Lie superalgebra of degree “ m, n ”. Subalgebras of the general linear Lie superalgebras are called linear Lie superalgebras. The orthosymplectic Lie superalgebra $\mathfrak{osp}(m|n)$ is a linear Lie superalgebra.

The bracket-preserving linear maps $\phi: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ are called representations of $\mathfrak{g}$.

Universal property characterization of universal enveloping algebras



The question: Is there a most general associative superalgebra in which $\mathfrak{g}$ embeds? The answer: The pair of the universal enveloping Lie superalgebra $U(\mathfrak{g})$ and the embedding ι from the tensor algebra construction. Note: $U(\mathfrak{g}) = T(\mathfrak{g}) / \langle xy - (-1)^{|x||y|} yx - [x, y] \rangle$.

Reduction algebra techniques using the simple Lie superalgebra $\mathfrak{osp}(1|2)$

The $(3|2)$ -dimensional Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1|2)$ has a matrix representation in $\mathbb{C}^{12}$:

$$\mathfrak{osp}(1|2)_{\bar{0}} \oplus \mathfrak{osp}(1|2)_{\bar{1}} = \mathbb{C}x_{-2\alpha} \oplus \mathbb{C}h \oplus \mathbb{C}x_{2\alpha} \bigoplus \mathbb{C}x_{-\alpha} \oplus \mathbb{C}x_{\alpha},$$

with

$$\begin{aligned} x_{-2\alpha} &= E_{23}, \\ x_{-\alpha} &= E_{13} + E_{21}, \\ h &= E_{22} - E_{33}, \\ x_{\alpha} &= E_{12} + E_{31}, \\ x_{2\alpha} &= E_{32}, \end{aligned}$$

for matrices E_{ij} having all zeroes except 1 in the (i, j) -entry.

Let $\mathfrak{g}_{\pm} = \mathbb{C}x_{\pm 2\alpha} \oplus \mathbb{C}x_{\pm \alpha}$ and $\mathfrak{h} = \mathbb{C}h$. A triangular decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{g}_{-} \oplus \mathfrak{h} \oplus \mathfrak{g}_{+}$.

Associative superalgebra homomorphism

There exists an embedding $\mathfrak{g} \hookrightarrow \mathcal{A} = U(\mathfrak{osp}(2|2))$. Therefore we have a map $\varphi: U(\mathfrak{g}) \rightarrow \mathcal{A}$ of unital associative superalgebras. Thus $\mathcal{A}$ is a $U(\mathfrak{g})$ -bimodule via left and right multiplication.

More details: Write $\{\alpha, \beta, \alpha + \beta\}$ for the set of positive roots of $\mathfrak{osp}(2|2)$. Set $x_{\alpha} = \frac{1}{\sqrt{2}}(x_{\alpha} + x_{\beta})$; $x_{-\alpha} = -\frac{1}{\sqrt{2}}(x_{-\alpha} + x_{-\beta})$; and, $h_{\alpha} = [x_{\alpha}, x_{-\alpha}]$. We recognize $\mathfrak{g}$ as the Lie subsuperalgebra of $\mathfrak{osp}(2|2)$ generated by x_{α} and $x_{-\alpha}$. Set $x_{\beta} = \frac{1}{\sqrt{2}}(x_{\alpha} - x_{\beta})$; $x_{-\beta} = -\frac{1}{\sqrt{2}}(x_{\alpha} - x_{\beta})$; and, $h_{\beta} = -\frac{1}{2}(h_{\alpha} - h_{\beta})$. Fix $V = \text{span}(\{x_{\beta}, h_{\beta}, x_{-\beta}\})$. Then $\mathfrak{osp}(2|2) = \mathfrak{g} \oplus V$ as $\mathfrak{osp}(1|2)$ -modules, where $V \cong \mathbb{C}^{1|2}$.

Universal Verma module (with respect to $\mathfrak{g}_{+}$)

Let $I_{+} = \mathcal{A}\mathfrak{g}_{+} = \mathcal{A}x_{\alpha}$ be the left ideal of $\mathcal{A}$ generated by the action of $\mathfrak{g}_{+}$ acting on $\mathcal{A}$ from the right. The left $\mathcal{A}$ -module $\mathcal{A}/I_{+}$ is the universal Verma module.

Nonlocalized reduction algebra [Mic73; Zhe06]

Let $N_{+} = N_{\mathcal{A}}(I_{+})$ be the normalizer of I_{+} , that is, the largest subalgebra of $\mathcal{A}$ in which I_{+} is a two-sided ideal. Then the subspace $(\mathcal{A}/I_{+})^{+}$ of $\mathfrak{g}_{+}$ -invariants within $\mathcal{A}/I_{+}$ is the quotient algebra N_{+}/I_{+} , which we call the nonlocalized reduction algebra associated to φ .

Sketch of proof: $x(a + I_{+}) = xa + I_{+} = 0 + I_{+}$ for all $x \in \mathfrak{g}_{+}$ if and only if $xa \in I_{+}$ for all $x \in \mathfrak{g}_{+}$ if and only if $a \in N_{+}$.

Importance of reduction algebra

Let $V^{+} \subset V$ be the $\mathfrak{g}_{+}$ -invariants. Since $I_{+}V^{+} \subset \{0\}$ and $N_{+}V^{+} \subset V^{+}$, we have the action $(n + I)v = nv + I$. So V^{+} is a left module over the reduction algebra and we can generate $\mathfrak{g}$ -submodules of V :

$$U(\mathfrak{g})(n + I_{+})v \subset V \text{ for } n \in N.$$

Normalizers may be hard to compute systematically

Set $\mathfrak{h} = \mathbb{C}h_{\alpha}$. Assume our spaces are localized (script font) with respect to a submonoid of $U(\mathfrak{h})$ generated by integral shifts of h_{α} . Let $\mathcal{I}_{-} = \mathfrak{g}_{-}\mathcal{A}$. We consider the double coset space $\mathcal{A}/\Pi := \mathcal{I}_{-}\backslash\mathcal{A}/\mathcal{I}_{+}$. There exists a unique element P of a certain completion of $U(\mathfrak{g})$ that induces a projection P_V on each V ; moreover, we have

$$P_{\mathcal{A}/\mathcal{I}_{+}}: \mathcal{A}/\mathcal{I}_{+} \rightarrow \mathcal{A}/\Pi \rightarrow \mathcal{N}_{+}/\mathcal{I}_{+} \rightarrow \mathcal{A}/\mathcal{I}_{+} \text{ with}$$

1. $\mathfrak{g}_{+}P = 0$;
2. $P\mathfrak{g}_{-} = 0$;
3. $[h, P] = 0$;
4. $P \equiv 1 \pmod{\Pi}$; and,
5. $\mathcal{A}/\Pi$ is a superalgebra under the diamond product.

Extremal projector

We use the Pochhammer symbol defined by

$$(x)_{\ell} = x(x+1)(x+2) \cdots (x+(\ell-1))$$

to write

$$Q_{\ell}(h_{\alpha}) = \frac{(-1)^{\ell}}{[\frac{\ell}{2}]!(h+1)_{[\frac{\ell+1}{2}]}}.$$

The extremal projector for $\mathfrak{g}$ is then

$$P = P_{\mathfrak{g}} = \sum_{\ell=0}^{\infty} Q_{\ell}(h_{\alpha})x_{-A}^{\ell}x_A^{\ell}.$$

Diamond product $\diamond$

Write $\bar{a} = a + \Pi$ for the double coset containing a for each a in $\mathcal{A}$. Then $\bar{a} \diamond \bar{b} := aPb + \Pi$. We use the monomial order $\overline{x_{-B}} < \overline{h_B} < \overline{x_B}$ in the diamond product table:

$\diamond$	$g(h_A)$	$\overline{x_{-B}}$	$\overline{h_B}$	$\overline{x_B}$
$f(h_A)$	$fg(h_A)$	$f(h_A)\overline{x_{-B}}$	0	$f(h_A)\overline{x_B}$
$\overline{x_{-B}}$	$g(h_A+1)\overline{x_{-B}}$	0	$\overline{x_{-B}} \diamond \overline{h_B}$	0
$\overline{h_B}$	$g(h_A)\overline{h_B}$	$\frac{h_A+2}{h_A+1}\overline{x_{-B}} \diamond \overline{h_B}$	$\overline{h_B} \diamond \overline{h_B}$	$\overline{h_B} \diamond \overline{x_B}$
$\overline{x_B}$	$g(h_A-1)\overline{x_B}$	T	$\frac{h_A+1}{h_A}\overline{h_B} \diamond \overline{x_B}$	0
$T = h_A - \frac{1}{h_A}\overline{h_B} \diamond \overline{h_B} - \frac{h_A+2}{h_A+1}\overline{x_{-B}} \diamond \overline{x_B}$				

Corank- n superderivation superalgebras

The Lie superalgebra $\mathfrak{osp}(1|2)$ also embeds in $\mathfrak{osp}(1|4)$, a rank-two Lie superalgebra distinct from $\mathfrak{osp}(2|2)$. Additionally, there is a chain of embeddings $\mathfrak{osp}(1|2) \hookrightarrow \mathfrak{osp}(2|2) \cong \mathfrak{vect}(0|2) \hookrightarrow \mathfrak{vect}(0|n+1)$ of $\mathfrak{osp}(1|2)$ inside vectorial Lie superalgebras, particularly, the derivation superalgebra of the exterior algebra on $n+1$ generators. Hence we continue the study of super representation theory and corank- n orthosymplectic reduction superalgebras relative to superderivations.

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References

- [HW22] J. T. Hartwig and D. A. Williams II. “Diagonal reduction algebra for $\mathfrak{osp}(1|2)$ ”. *Theoretical and Mathematical Physics* 210.2 (Feb. 2022), pp. 155–171. ISSN: 0040-5779, 1573-9333. DOI: 10.1134/S0040577922020015.
- [Kho04] S. M. Khoroshkin. “Extremal Projector and Dynamical Twist”. *Theoretical and Mathematical Physics* 139.1 (Apr. 2004), pp. 582–597. ISSN: 0040-5779. DOI: 10.1023/B:TAMP.0000022749.42512.f.d.
- [Mic73] J. Mickelsson. “Step Algebras of Semi-Simple Subalgebras of Lie Algebras”. *Reports on Mathematical Physics* 4.4 (June 1973), pp. 307–318. ISSN: 00344877. DOI: 10.1016/0034-4877(73)90006-2.
- [Rie16] E. Riehl. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Mineola, New York: Dover Publications, 2016. 240 pp. ISBN: 978-0-486-80903-8.
- [Wal24] C. Walton. *Symmetries of Algebras*. Vol. 1. Oklahoma City, OK: 619 Wreath Publishing, 2024. ISBN: 978-1-958469-21-7.
- [Zhe06] D. P. Zhelobenko. *Principal Structures and Methods of Representation Theory*. Translations of Mathematical Monographs v. 228. Providence, R.I: American Mathematical Society, 2006. 430 pp. ISBN: 978-0-8218-3731-3.